On τ-functions of Zakharov-Shabat and other matrix hierarchies of integrable equations

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Abstract

Matrix hierarchies are: multi-component KP, general Zakharov-Shabat (ZS) and its special cases, e.g., AKNS. The ZS comprises all integrable systems having a form of zero-curvature equations with rational dependence of matrices on a spectral parameter. The notion of a τ-function is introduced here in the most general case along with formulas linking τ-functions with wave Baker functions. The method originally invented by Sato et al. for the KP hierarchy is used. This method goes immediately from definitions and does not require any assumption about the character of a solution, being the most general. Applied to the matrix hierarchies, it involves considerable sophistication. The paper is self-contained and does not expect any special prerequisite from a reader.

1. Introduction.

Integrable systems of differential equations exist not isolated but united in large communities called hierarchies. All equations inside a hierarchy are commuting with each other. The first known hierarchies were generalized Korteweg-de Vries (KdV) hierarchies, one for every natural number n. (For detail, see, e.g., [9]). Then an immense Kadomtsev-Petviashvili (KP) hierarchy was found which united all the KdV's. Those hierarchies consisted of scalar equations. Almost immediately they were generalized to matrix equations. They formed "multi-component" KdV's and KP.

All the above mentioned hierarchies are generated by linear differential (KdV) or pseudo-differential (KP) operators of arbitrary orders. Equations of another type are generated by matrix first order differential operators linearly depending on a spectral parameter. These are AKNS (for Ablowitz, Kaupp, Newell and Segur) with 2×2 matrix first order operators, they were generalized by Dubrovin to $n \times n$ matrices; we call the latter AKNS-D hierarchies. The next generalization is when linear operators depend on a parameter as polynomials of any degree. Finally, the most general case involves arbitrary rational dependence on a parameter. These equations are called general Zakharov-Shabat (ZS) equations. They also form a hierarchy (see [7]). The hierarchy with polynomial dependence on a parameter is a special case of the general ZS when there is a single pole, at infinity; we call this hierarchy s-p ZS. All KdV's and AKNS's are nothing but reductions of the general ZS hierarchy. The exact definitions will be given below.

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The importance of the theory of integrable system was essentialy enhanced with the invention of the "tau"-function by mathematicians of the Kyoto school, see [1], [2]. This one single function of infinitely many "time" variables replaces infinitely many dynamical variables, coefficients of linear differential or pseudo-differential operators. It happened that this function linked integrable systems to Lie algebras representations and to many problems of modern physics, such as conformal field theory, matrix models in the statistical physics, 2-dimensional gravity and string theory. Up to now, all these achievements applied solely to scalar hierarchies, nth KdV and KP. There are also some published results about the multi-component KP. Concerning the general ZS eqations and their τ-functions, we know the only work [10] done in very abstract terms; it is difficult to extract concrete formulas from it. The aim of the present article is to fill up this gap. Physicists have not turned yet their attention to the general ZS hierarchy (except some particular equations of these hierarchy). We believe that its time will come sooner or later.

The paper is self-contained and, formally speaking, does not require a special prerequisite (see also [9]). It was easier not to start with the most complicated case of the general ZS but to pass gradually from the simplest to the most difficult model referring when needed to what was proven before.

In the first part of the paper we deal with the multi-component KP (mcKP). It is defined in [1]. In [3] and [4] there are formulas written for its τ-function, in both the articles without proofs. Therefore it is difficult to guess what was the way they followed. Most probably, they used the techniques of free fermion representations. Meanwhile, those authors had suggested their own excellent method which was invented in [2] for KP, based on nothing but the bilinear identity, i.e., being close to very first definitions. The advantage of this approach is its full generality, independence of the origin and the nature of a solution. Our first goal was to adjust this method to the mcKP hierarchy. Basically, the method remains the same as in [2], however, it becomes a little tricky. (In [5] we derived the τ-function in terms of the Grassmannian, in [6] found it for special, algebraic geometrical solutions; in contrast to that, we discuss now the general case).

The next part is devoted to the simplest special case of the ZS hierarchy, namely, the single-pole hierarchy (s-p ZS). It is closely connected with the mcKP since it is proven below that a Baker function of the s-p ZS is at the same time that of mcKP, and the s-p ZS is a subhierarchy of the mcKP. A similar statement was made before in [5].

Then we introduce a "not normalized" s-p ZS hierarchy which differs from the previous one by the fact that the expansion of its Baker function in powers of a spectral parameter starts with a matrix of a general form, not from the unity. It can be reduced to the normalized ZS. Nevertheless, it is convenient to study this case separately because it provides a good preparation for the general ZS where one cannot normalize Baker functions simultaneously at all poles.

Finally, and this is the main point, we treat the general ZS hierarchy. It is discussed in [7] in what sense one can understand the totality of all ZS equations as a hierarchy, i.e., as a set of commuting vector fields. There are definitions of a Baker function, of a corresponding Grassmannian, etc. in that paper. However, it is lacking a concept of the τ-function. We are doing this now. The main results of the present article are contained in the theorems of sect. 4 and sect. 7 and the proposition 3 and its corollary of sect. 5.

Despite the absence of a general definition and of a proof of the existence of the τ-function, there were a few examples of that function found earlier. In [7] this is done for soliton-type solutions and quite recently, in [8], for algebraic geometrical solutions that can be expressed in terms of θ-functions. Those examples were stimulating for the present study.

2. Multi-component KP.

Let

$$L = A\partial + u_0 + u_1\partial^{-1} + \cdots, \quad \partial = d/dx$$

be a pseudo-differential operator where u_i are $n \times n$ matrices, $A = \mathrm{diag}(a_1, ..., a_n)$, a_i are distinct non-zero constants. Diagonal elements of u_0 are assumed to be zero.

Let $R_\alpha = \sum_{j=0}^\infty R_{j\alpha}\partial^{-j}$, $\alpha = 1, ..., n$, where $R_{0\alpha} = E_\alpha$, E_α is a matrix having only one non-zero element on the (α, α) place which is equal to 1; R_α is supposed to satisfy

$$[L, R_\alpha] = 0.$$

It is shown below that such matrices exist being

$$R_\alpha R_\beta = \delta_{\alpha\beta}R_\alpha, \quad \sum_{\alpha=1}^n R_\alpha = I$$

(i.e. this is a spectral decomposition of the unity). The mcKP hierarchy (multi-component KP) is

$$\partial_{k\alpha}L = [(L^k R_\alpha)_+, L], \ \partial_{k\alpha}R_\beta = [(L^k R_\alpha)_+, R_\beta] \ \partial_{k\alpha} = \partial/\partial t_{k\alpha}, \quad k = 0, 1, ...; \ \alpha = 1, ..., n$$

and $t_{k\alpha}$ are the "time variables" of the hierarchy. The subscript $+$ refers, as usual, to a purely differential part of a pseudo-differential operator, $(\sum a_k \partial^k)_+ = \sum_{k \geq 0} a_k \partial^k$, $A_- = A - A_+$.

It can be shown that the equations for different k, α commute. The variables x and $t_{k\alpha}$ are not independent:

$$\partial = \sum_\alpha a_\alpha^{-1}\partial_{1\alpha}$$

(Greek indices always run from 1 to n).

Let

$$L = \hat{w}A\partial\hat{w}^{-1}, \quad \text{where} \ \ \hat{w} = \hat{w}(A\partial) = \sum_0^\infty w_i(A\partial)^{-i}, \ w_0 = I;$$

Then $R_\alpha = \hat{w}E_\alpha\hat{w}^{-1}$ has all needed properties. Put

$$w = \hat{w}(A\partial)\exp\xi(t, z) = \hat{w}(z)\exp\xi(t, z); \quad \text{where} \ \ \xi(t, z) = \sum_{k=0}^\infty \sum_{\alpha=1}^n z^k E_\alpha t_{k\alpha}.$$

This is the *Baker function*; it satisfies the equations

$$Lw = zw, \quad \text{and} \ \ \partial_{k\alpha}w = (L^k R_\alpha)_+ w.$$

The latter equation is equivalent to

$$\partial_{k\alpha}\hat{w} = -(L^k R_\alpha)_-\hat{w}.$$

Remark 1. It is very important to note that the series $\hat{w}$ are defined up to a multiplication on the right by series $\sum_0^\infty c_i \partial^{-i}$ with constant diagonal matrices c_i where $c_0 = I$. Correspondingly, the Baker function is defined up to a multiplication by $\sum_0^\infty a_i z^{-i}$. Two functions which differ by such a factor are said to be equivalent. For two equivalent Baker functions the Lax operator L is the same. All the formulas below will be obtained up to the equivalence.

We have $\partial_{0\alpha}\hat{w} = -(R_\alpha - E_\alpha)\hat{w} = -\hat{w}E_a + E_\alpha\hat{w} = [E_\alpha, \hat{w}]$. Symmetries related to "zero" time variables $t_{0\alpha}$ are similarity transformations with constant matrices.

The *adjoint Baker function* is

$$w^a = (\hat{w}^*(A\partial))^{-1}\exp(-\xi(t,z))$$

where the star means the conjugation: for every matrix X the equality $(X\partial)^* = -\partial X^*$ holds where X^* is the transpose of X.

The equations

$$L^*w^a = zw^a, \quad \text{and} \quad \partial_{k\alpha}w^a = -(L^k R_\alpha)^*_+ w^a$$

hold.

Remark 2. Our definition of the mcKP differs from that in [1],[2] and [3] where $u_0 = 0$ and $A = I$. It is easy to show that in our definition the coefficients of the equations are local in terms of u_i's, i.e., differential polynomials of them. Indeed, the dressing formula $L = \hat{w}A\partial\hat{w}^{-1}$ permits to express every differential polynomial in elements of w_i's as a differential polynomial in elements of u_i's which is also an ordinary polynomial in w_i's (i.e., it does not depend on derivatives of w_i's). Then, the elements of R_α's are such polynomials, too. Let us show that, in fact, they do not depend on w_i's at all. Let us give to $\hat{w}$ an infinitesimal deformation $\delta\hat{w}$ such that L is not changed. This means that $\delta L = [\delta\hat{w}\cdot\hat{w}^{-1}, A\partial] = 0$. This easily implies that the matrix $K = \delta\hat{w}\cdot\hat{w}^{-1}$ is constant and diagonal. Now, $\delta R_\alpha = [K, E_\alpha] = 0$. The rest is clear. The fact that all diagonal elements of A are distinct is crucial. It is easy to compute that otherwise R_α are not local. If one is only interested in the hierarchy in terms of Baker functions, not of the operator L, then this distinction is not important.

The significance of the mcKP, as well as KP, is in their universality.

Proposition. Universality of the mcKP hierarchy. *If an expression of the form*

$$w = \hat{w}(A\partial)\exp\xi(t,z) = \hat{w}(z)\exp\xi(t,z); \quad \text{where} \quad \hat{w}(A\partial) = \sum_0^\infty w_i(A\partial)^{-i}, \ w_0 = I$$

satisfies arbitrary equations $\partial_{k\alpha}w = \overline{B}_{k\alpha}w$ with some differential operators $\overline{B}_{k\alpha}$ then this is nothing but mcKP.

Indeed, the given equations yield

$$0 = \partial_{k\alpha}\hat{w}\cdot e^\xi + \hat{w}E_\alpha z^k e^\xi - \overline{B}_{k\alpha}w = \partial_{k\alpha}\hat{w}\cdot e^\xi + \hat{w}E_\alpha(A\partial)^k e^\xi - \overline{B}_{k\alpha}w.$$

Letting $L = \hat{w}A\partial\hat{w}^{-1}$ and $R_\alpha = \hat{w}E_\alpha\hat{w}^{-1}$ we have $\partial_{k\alpha}\hat{w}\cdot\hat{w}^{-1} + R_\alpha L^k - \overline{B}_{k\alpha} = 0$. Taking the positive part of this equation, we get $\overline{B}_{k\alpha} = (R_\alpha L^k)_+$ and the negative part is

$\partial_{k\alpha}\hat{w} = -(R_\alpha L^k)_-\hat{w}$. This is the equation of the hierarchy. $\square$

3. Bilinear identity.

The so-called bilinear identity is basic for Sato's theory.

Lemma. *Let* $\Phi = \sum \Phi_i(A\partial)^i$ *and* $\Psi = \sum \Psi_i(A\partial)^i$ *be two pseudo-differential operators* (ΨDO)*. Then the equality*

$$\mathrm{res}_\partial \Phi\Psi^* = \mathrm{res}_z(\Phi e^\xi)A^{-1}(\Psi e^{-\xi})^*$$

holds.

The notations res_∂ and res_z mean, as usual, coefficients of ∂^{-1} and z^{-1}.

Proof. It is easy to check that both the left- and the right-hand side are equal to $\sum \Phi_i A^{-1}\Psi^*_{-i-1}(-1)^{i+1}$. $\square$

Proposition. *If* $\Phi = \sum \Phi_i(A\partial)^i$ *is a* ΨDO*, and* $w = \Phi \exp \xi(t, z)$*,* $w^a = (\Phi^*)^{-1}\exp(-\xi(t, z))$ *then*

$$\mathrm{res}_z(\partial^i w)A^{-1}(w^a)^* = 0. \tag{1a}$$

Moreover, if w *depends on infinitely many variables* $t_{i\alpha}$*,* $i = 0, 1, ..., \alpha = 1, ..., n$ *and satisfies a system of differential equations of the form* $\partial_{i\alpha}w = B_{i\alpha}w$ *where* $B_{i\alpha}$ *are any differential (matrix) operators in* $\partial = \sum a_\alpha^{-1}\partial_{1\alpha}$ *then*

$$\mathrm{res}_z(\partial_{i_1\alpha_1}\partial_{i_2\alpha_2}...\partial_{i_s\alpha_s}w)A^{-1}(w^a)^* = 0 \tag{1b}$$

for an arbitrary set of indices $i_1, \alpha_1, i_2, \alpha_2, ..., i_s, \alpha_s$*. This happens, e.g., when* w *is a Baker function of the mcKP hierarchy.*

Conversely, if there are two expressions of the form $w = \sum_0^\infty w_i(t, z)z^{-i}\exp\xi$ *and* $w^a = \sum v_i(t, z)z^{-i}\exp(-\xi)$ *with* $w_0 = v_0 = I$*, and Eq.(1a) holds for them, then letting* $\Phi = \sum w_i(A\partial)^{-i}$ *we will have* $w = \Phi\exp\xi$ *and* $w^a = (\Phi^*)^{-1}\exp(-\xi)$*.*

Moreover, if the stronger equality (1b) holds, then w *and* w^a *are the Baker and the adjoint Baker functions of the mcKP.*

Proof. We have

$$\mathrm{res}_z(\partial^i w)A^{-1}(w^a)^* = \mathrm{res}_z(\partial^i \Phi e^\xi)A^{-1}((\Phi^*)^{-1}e^{-\xi})^*$$

$$= \mathrm{res}_\partial \partial^i \Phi((\Phi^*)^{-1})^* = \mathrm{res}_\partial \partial^i \Phi\Phi^{-1} = \mathrm{res}_\partial \partial^i = 0.$$

This proves the first statement. Now, the equations $\partial_{i\alpha}w = B_{i\alpha}w$ allow to express all the derivatives $\partial_{i\alpha}$ in terms of ∂ and then to apply (1a) which proves (1b).

The first statement of the converse proposition can be obtained in the following way. Let $\Phi = \sum w_i(A\partial)^{-i}$ and $\Psi = \sum v_i(-A\partial)^{-i}$ then $w = \Phi\exp\xi$ and $w^a = \Psi\exp(-\xi)$. We have

$$0 = \mathrm{res}_z(\partial^i \Phi e^\xi)A^{-1}(\Psi e^{-\xi})^* = \mathrm{res}_\partial \partial^i \Phi\Psi^*$$

for all $i \geq 0$. The operator $\Phi\Psi^*$ is $I + O(\partial^{-1})$, and the last equality implies that the negative part is zero. Hence $\Phi\Psi^* = I$ and $\Psi = (\Phi^*)^{-1}$.

Now, let (1b) hold. Put $L = \Phi A \partial \Phi^{-1}$. We have

$$((\partial_{k\alpha}\Phi) + (L^k R^\alpha)_- \Phi)e^\xi = (\partial_{k\alpha} \cdot \Phi - \Phi(A\partial)^k E_\alpha + (L^k R^\alpha)_- \Phi)e^\xi = (\partial_{k\alpha} - (L^k R^\alpha)_+)\Phi e^\xi.$$

Then, applying the assumption and the lemma,

$$0 = \mathrm{res}_z \partial^i (\partial_{k\alpha} - (L^k R^\alpha)_+) w A^{-1}(w^a)^* = \mathrm{res}_z \partial^i (\partial_{k\alpha} - (L^k R^\alpha)_+)\Phi e^\xi A^{-1}((\Phi^*)^{-1} e^{-\xi})^*$$

$$= \mathrm{res}_\partial \partial^i ((\partial_{k\alpha}\Phi) + (L^k R^\alpha)_- \Phi)\Phi^{-1}.$$

This yields $(\partial_{k\alpha}\Phi) + (L^k R^\alpha)_- \Phi = 0$, i.e., the equation of the hierarchy. $\square$

The bilinear identity can also be written in a dual form

$$\mathrm{res}_z w A^{-1}(\partial_{i_1\alpha_1}\partial_{i_2\alpha_2}...\partial_{i_s\alpha_s} w^a)^* = 0.$$

The proof is similar.

Very often they use the identity in the form

$$\mathrm{res}_z w(t, z) A^{-1}(w^a(t', z))^* = 0$$

where t' is another set of values $t_{k\alpha}$. This identity makes sense as a formal expansion in powers of $t'_{k\alpha} - t_{k\alpha}$.

4. τ-function.

Let $G_\alpha(\zeta)$ be an operator of translation acting as

$$G_\alpha(\zeta) f(t, z) = f(..., t_{k\gamma} - \delta_{\alpha\gamma}\frac{1}{k\zeta^k}, ..., z).$$

Let

$$N_\alpha(\zeta) = -\sum_{j=0}^\infty \zeta^{-j-1}\partial_{j\alpha} + \partial_\zeta, \quad \partial_\zeta = \partial/\partial\zeta.$$

It is easy to see that $N_\alpha(\zeta)G(\zeta)f(t, z) = 0$.

According to the bilinear identity,

$$\mathrm{res}_z w(t, z) A^{-1} G_\beta(\zeta)(w^a(t, z))^* = 0.$$

We have

$$G_\beta(\zeta) \exp(-\sum_{k\gamma} t_{k\gamma} E_\gamma z^k) = (I - E_\beta + (1 - \frac{z}{\zeta})^{-1} E_\beta) \exp(-\sum_{k\gamma} t_{k\gamma} E_\gamma z^k)$$

as it is easy to check. If $w(z) = \hat{w}(z) \exp\xi$ and $w^a(z) = \hat{w}^a(z) \exp(-\xi)$ then

$$\mathrm{res}_z \hat{w}(z) A^{-1}(I - E_\beta + (1 - \frac{z}{\zeta})^{-1} E_\beta) G_\beta(\zeta)(\hat{w}^a(z))^* = 0.$$

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It is easy to see that if $f(z) = \sum f_i z^i$ then $\operatorname{res}_z f(z)(1 - z/\zeta)^{-1} = \zeta f_-(\zeta)$ where the subscript "$-$" symbolizes the negative part of the series. We have $\hat{w} = I + w_1 z^{-1} + \dots$ and, as a simple calculation shows, $(\hat{w}^a)^* = I - A w_1 A^{-1} z^{-1} + \dots$. The identity becomes

$$w_1(I - E_\beta)A^{-1} - (I - E_\beta)G_\beta w_1 A^{-1} + \zeta[\hat{w}(\zeta)A^{-1}E_\beta G_\beta(\zeta)(\hat{w}^a(\zeta))^*]_- = 0.$$

The (β, β)th element of this matrix identity is $\hat{w}_{\beta\beta}(\zeta)a_\beta^{-1}G_\beta(\zeta)(\hat{w}^a(\zeta))^*_{\beta\beta} - a_\beta^{-1}I = 0$. Thus, we have

$$\hat{w}_{\beta\beta}(\zeta)G_\beta(\zeta)(\hat{w}^a(\zeta))_{\beta\beta} = I. \tag{2}$$

The shifted $(\hat{w}^a(\zeta))_{\beta\beta}$ happens to be just the inverse of $\hat{w}_{\beta\beta}(\zeta)$.

Let us take now the (α, β)th element of the matrix identity:

$$-a_\beta^{-1}G_\beta(\zeta)w_{1,\alpha\beta} + \zeta\hat{w}_{\alpha\beta}(\zeta)a_\beta^{-1}G_\beta(\zeta)(\hat{w}^a(\zeta))_{\beta\beta} = 0.$$

Using (2), transform this to

$$G_\beta(\zeta)w_{1,\alpha\beta} = \zeta\hat{w}_{\alpha\beta}(\zeta)(\hat{w}_{\beta\beta}(\zeta))^{-1}. \tag{3}$$

Now, consider a more complicated relation which also follows from the bilinear identity:

$$\operatorname{res}_z w(z)A^{-1}G_\alpha(\zeta_1)G_\beta(\zeta_2)(w^a(z))^* = 0.$$

In the case when $\alpha = \beta$ this reduces to

$$\operatorname{res}_z \hat{w}(z)A^{-1}[I - E_\beta + (1 - \frac{z}{\zeta_1})^{-1}(1 - \frac{z}{\zeta_2})^{-1}E_\beta]G_\beta(\zeta_1)G_\beta(\zeta_2)(\hat{w}^a(z))^* = 0.$$

Taking the (β, β)th element we have

$$\operatorname{res}_z \hat{w}_{\beta\beta}(z)(1 - \frac{z}{\zeta_1})^{-1}(1 - \frac{z}{\zeta_2})^{-1}G_\beta(\zeta_1)G_\beta(\zeta_2)(\hat{w}^a(z))_{\beta\beta} = 0$$

or

$$\operatorname{res}_z \hat{w}_{\beta\beta}(z)[\zeta_1^{-1}(1 - \frac{z}{\zeta_1})^{-1} - \zeta_2^{-1}(1 - \frac{z}{\zeta_2})^{-1}]G_\beta(\zeta_1)G_\beta(\zeta_2)(\hat{w}^a(z))_{\beta\beta} = 0$$

which yields

$$\hat{w}_{\beta\beta}(\zeta_1)G_\beta(\zeta_1)G_\beta(\zeta_2)(\hat{w}^a(\zeta_1))^* = \hat{w}_{\beta\beta}(\zeta_2)G_\beta(\zeta_1)G_\beta(\zeta_2)(\hat{w}^a(\zeta_2))_{\beta\beta}.$$

Using (2), we obtain

$$\frac{G_\beta(\zeta_2)\hat{w}_{\beta\beta}(\zeta_1)}{\hat{w}_{\beta\beta}(\zeta_1)} = \frac{G_\beta(\zeta_1)\hat{w}_{\beta\beta}(\zeta_2)}{\hat{w}_{\beta\beta}(\zeta_2)}.$$

Taking a logarithm and denoting $\ln \hat{w} = f$ we get

$$(G_\beta(\zeta_2) - 1)f_{\beta\beta}(\zeta_1) = (G_\beta(z_1) - 1)f_{\beta\beta}(\zeta_2). \tag{4}$$

In the case when $\alpha \neq \beta$ the identity is

$$\operatorname{res}_z \hat{w}(z)A^{-1}[I - E_\alpha - E_\beta + (1 - \frac{z}{\zeta_1})^{-1}E_\alpha + (1 - \frac{z}{\zeta_2})^{-1}E_\beta]G_\alpha(\zeta_1)G_\beta(\zeta_2)(\hat{w}^a(z))^* = 0.$$

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The (α, α)th element of this matrix identity is

$$\zeta_1 \hat{w}_{\alpha\alpha}(\zeta_1) a_\alpha^{-1} G_\alpha(\zeta_1) G_\beta(\zeta_2)(\hat{w}^a(\zeta_1))_{\alpha\alpha} + \zeta_2 \hat{w}_{\alpha\beta}(\zeta_2) a_\beta^{-1} G_\alpha(\zeta_1) G_\beta(\zeta_2)(\hat{w}^a(\zeta_2))_{\beta\alpha}^* - \zeta_1 a_\alpha^{-1} I = 0.$$

The (β, α)th element is

$$\zeta_2 \hat{w}_{\beta\beta}(\zeta_2) a_\beta^{-1} G_\alpha(\zeta_1) G_\beta(\zeta_2)(\hat{w}^a(\zeta_2))_{\beta\alpha}^* + \zeta_1 \hat{w}_{\beta\alpha}(\zeta_1) a_\alpha^{-1} G_\alpha(\zeta_1) G_\beta(\zeta_2)(\hat{w}^a(\zeta_1))_{\alpha\alpha} = 0.$$

Eliminating $(\hat{w}^a)_{\beta\alpha}^*$ from two equations and applying (2), we obtain

$$-\hat{w}_{\beta\beta}(\zeta_2) + (\hat{w}_{\alpha\alpha}(\zeta_1)\hat{w}_{\beta\beta}(\zeta_2) - \hat{w}_{\beta\alpha}(\zeta_1)\hat{w}_{\alpha\beta}(\zeta_2))G_\beta(\zeta_2)(\hat{w}_{\alpha\alpha}(\zeta_1))^{-1} = 0.$$

Take a logarithm:

$$\ln \hat{w}_{\beta\beta}(\zeta_2) = \ln(\hat{w}_{\alpha\alpha}(\zeta_1)\hat{w}_{\beta\beta}(\zeta_2) - \hat{w}_{\beta\alpha}(\zeta_1)\hat{w}_{\alpha\beta}(\zeta_2)) - G_\beta(\zeta_2) \ln \hat{w}_{\alpha\alpha}(\zeta_1)$$

and subtract this equation from one obtained by permutation of α and β, ζ_1 and ζ_2. The result is

$$(G_\beta(\zeta_2) - 1)f_{\alpha\alpha}(\zeta_1) = (G_\alpha(\zeta_1) - 1)f_{\beta\beta}(\zeta_2), \quad f = \ln \hat{w}. \tag{5}$$

Eq.(4) is a special case of this one when $\alpha = \beta$.

Now we have to prove the existence of a function $\tau(t)$ such that $f_{\alpha\alpha}(\zeta) = (G_\alpha(\zeta)-1)\ln\tau$. If the operator $(G_\alpha(\zeta) - 1)$ had an inverse, this would immediately follow from (5). This operator has a kernel consisting of constants (with respect to $\{t_{k\alpha}\}$). Let us apply the operator $N_\alpha(\zeta_1)$ to Eq.(5):

$$G_\beta(\zeta_2)N_\alpha(\zeta_1)f_{\alpha\alpha}(\zeta_1) - N_\alpha(\zeta_1)f_{\alpha\alpha}(\zeta_1) = \sum_{j=0}^{\infty} \zeta_1^{-j-1}\partial_{j\alpha}f_{\beta\beta}(\zeta_2).$$

Then multiply this by ζ_1^i and take res_{ζ_1}:

$$b_{i\alpha} \equiv \mathrm{res}_{\zeta_1}\zeta_1^i N_\alpha(\zeta_1)f_{\alpha\alpha}(\zeta_1) = G_\beta(\zeta_2)\mathrm{res}_{\zeta_1}\zeta_1^i N_\alpha(\zeta_1)f_{\alpha\alpha}(\zeta_1) + \partial_{i\alpha}f_{\beta\beta}(\zeta_2),$$

i.e.,

$$b_{i\alpha} = G_\beta(\zeta_2)b_{i\alpha} + \partial_{i\alpha}f_{\beta\beta}(\zeta_2). \tag{6}$$

Here (i, α) is an arbitrary pair of indices, one can replace them by (j, γ):

$$b_{j\gamma} = G_\beta(\zeta_2)b_{j\gamma} + \partial_{j\gamma}f_{\beta\beta}(\zeta_2).$$

Differentiating the first equality with respect to $t_{j\gamma}$, the second with respect to $t_{i\alpha}$ and subtracting, we have $(G_\beta(\zeta_2) - 1)(\partial_{j\gamma}b_{i\alpha} - \partial_{i\alpha}b_{j\gamma}) = 0$ whence $\partial_{j\gamma}b_{i\alpha} - \partial_{i\alpha}b_{j\gamma}$ is a constant. It is not difficult to see from the definition of $b_{i\alpha}$ that this constant can be only zero. Thus, $\partial_{j\gamma}b_{i\alpha} = \partial_{i\alpha}b_{j\gamma}$. This implies the existence of a function of the variables $\{t_{i\alpha}\}$, we call it $\ln \tau(t)$, such that $b_{i\alpha} = \partial_{i\alpha}\ln\tau$:

$$\mathrm{res}_\zeta \zeta^i(-\sum_0^\infty z^{-j-1}\partial_{j\alpha} + \partial_\zeta)\ln\hat{w}_{\alpha\alpha}(\zeta) = \partial_{i\alpha}\ln\tau. \tag{7}$$

The equation (6) yields that $\partial_{i\alpha} f_{\beta\beta}(\zeta) = (G_\beta(\zeta) - 1)b_{i\alpha} = (G_\beta(\zeta) - 1)\partial_{i\alpha} \ln \tau$ and $f_{\beta\beta}(\zeta) = (G_\beta(\zeta) - 1) \ln \tau + $const. In more detail, this formula looks like this:

$$\hat{w}_{\beta\beta}(\zeta) = c_\beta(\zeta) \frac{\tau(..., t_{k\gamma} - \delta_{\beta\gamma} \cdot 1/(k\zeta^k), ...)}{\tau(t)}. \tag{8}$$

In the numerator only the variables $t_{k\gamma}$ with $\gamma = \alpha$ are shifted. The constant $c_\beta(\zeta)$ is a series $c_\beta(\zeta) = \sum_{i=0}^\infty c_{i\beta} z^{-i}$ with $c_{0\beta} = 1$.

We have obtained this formula only for diagonal elements of $\hat{w}$ yet. Eq.(7) is a conversion of Eq.(8). Let $C = $diag $c_\beta(\zeta)$, a constant diagonal matrix. Then the Baker function wC^{-1} is equivalent to w. For this function (8) holds with $c_\beta = 1$.

Let us return to Eq.(3). We find $\hat{w}_{\alpha\beta}(\zeta) = \zeta^{-1} G_\beta(\zeta) w_{1,\alpha\beta} \cdot \hat{w}_{\beta\beta}$, substituting $\hat{w}_{\beta\beta}$ from (8) and denoting

$$\tau_{\alpha\beta}(t) = \tau(t) w_{1,\alpha\beta}, \quad \alpha \neq \beta, \tag{9}$$

this becomes $\hat{w}_{\alpha\beta}(\zeta) = \zeta^{-1} G_\beta(\zeta) \tau_{\alpha\beta} \cdot (\tau(t))^{-1}$, or

$$\hat{w}_{\alpha\beta}(\zeta) = \zeta^{-1} c_\beta(\zeta) \frac{\tau_{\alpha\beta}(..., t_{k\gamma} - \delta_{\beta\gamma} \cdot 1/(k\zeta^k), ...)}{\tau(t)}, \quad \alpha \neq \beta. \tag{10}$$

Thus, only those variables $t_{k,\gamma}$ are shifted whose index γ coincides with the number of the column, β. Thus, we have a theorem:

Theorem. *For any Baker function there are functions $\tau(t)$ and $\tau_{\alpha\beta}(t)$ and constant series $c_\beta(\zeta)$ such that Eqs.(8) and (10) hold. Coefficients $c_\beta(\zeta)$ are insignificant if a Baker functions is considered to within the equivalence.*

The formulas (8) and (10) are the main formulas of the theory of the τ-function.

Remark. The definition of the τ-function and the derivation of the formulas (8) and (10) based on the bilinear identity does not depend on the property of the matrix A to have distinct elements on the diagonal. It remains valid even if $A = I$. This will be used in the next section.

5. Single-pole Zakharov-Shabat hierarchy.

The general Zakharov-Shabat equation is $[I\partial + U(z), I\partial_t + V(z)] = 0$ where matrices U and V are rational functions of a parameter z. In [7] it was explained in what sense the totality of all possible equations of this form can be considered as one hierarchy. Now we are interested in the case when both the functions $U(z)$ and $V(z)$ have a single pole which is at infinity, i.e., they are polynomials in z.

Let $\hat{w} = \sum_{i=0}^\infty w_i z^{-i}$ be a formal series, $w_0 = I$.

Definition. The single-pole ZS hierarchy is the totality of all the equations

$$\partial_{l\alpha} \hat{w} = -(z^l R_\alpha)_- \hat{w} \quad \text{where} \quad R_\alpha = \hat{w} E_\alpha \hat{w}^{-1}. \tag{11}$$

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The subscript "−" refers to the negative part of an expansion in powers of z.

Letting $w = \hat{w} \exp \xi(t, z)$, where ξ is as before, we get an equivalent form of the equations of the hierarchy

$$\partial_{l\alpha} w = B_{l\alpha} w, \quad B_{l\alpha} = (z^l R_\alpha)_+. \tag{12}$$

The same equation can also be expressed as

$$w \partial_{l\alpha} \cdot w^{-1} = \hat{w}(\partial_{l\alpha} - z^l E_\alpha)\hat{w}^{-1} = I\partial_{l\alpha} - B_{l\alpha}. \tag{13}$$

Thus, dressing of $I\partial_{l\alpha}$ yields a first-order differential operator (13) that is a lth degree polynomial in z. The expression w is called a *formal Baker function*.

It can be proven that the operators $\partial_{l\alpha}$ commute. This fact and Eq.(13) imply that operators $I\partial_{l\alpha} - B_{l\alpha}$ commute, i.e.,

$$\partial_{l\alpha} B_{m\beta} - \partial_{m\beta} B_{l\alpha} - [B_{l\alpha}, B_{m\beta}] = 0. \tag{14}$$

Let λ_l, $l = 0, ..., m+1$ be a sequence of constant diagonal matrices, $\lambda_l =$ diag $(\lambda_{l\alpha})$, $\lambda_{m+1,\alpha} = a_\alpha$ being distinct, and $\partial = -\sum_{l=0}^{m+1} \sum_{\alpha=1}^{n} \lambda_{l\alpha}\partial_{l\alpha}$. Set

$$L = -\sum_{l=0}^{m+1} \sum_{\alpha=1}^{n} \lambda_{l\alpha}(I\partial_{l\alpha} - B_{l\alpha}) = I\partial + U \tag{15}$$

where $U = \sum_{i=0}^{m+1} \sum_{\alpha=1}^{n} \lambda_{i\alpha} B_{l\alpha}$. Then

$$L = w\partial w^{-1} = I\partial + U = I\partial + u_0 + u_1 z + ... + u_m z^m - Az^{m+1}, \quad A = \text{diag } a_\alpha. \tag{16}$$

The hierarchy equations imply

$$\partial_{m\beta} L = [B_{m\beta}, L].$$

If M is another operator defined in the same way as L with other matrix diagonal coefficients, $\mu_{l\alpha}$ instead of $\lambda_{l\alpha}$, then $[L, M] = 0$. This is exactly the ZS equation with a single pole.

The notion of the *equivalence* is the same as for the mcKP: two Baker functions are equivalent if they differ by a factor on the right which is a constant diagonal matrix series. Then $B_{l\alpha}$'s remain the same along with all differential operators L.

Proposition 1. Universal property. *Let $\hat{w}$ be a series $\hat{w} = \sum_{i_0}^{\infty} w_i z^{-i}$, $w_0 = I$ and $w = \hat{w} \exp \xi$. All the functions depend on variables $t_{k\alpha}$. If w satisfies an equation of the form $\partial_{k\alpha} w = \overline{B}_{k\alpha} w$ where $\overline{B}_{k\alpha} w$ is a polynomial in z then this is an equation of the hierarchy, i.e. $\overline{B}_{k\alpha} = (z^k R_\alpha)_+$.*

Proof. We have

$$0 = \partial_{k\alpha}\hat{w} \cdot e^\xi + \hat{w}E_\alpha z^k e^\xi - \overline{B}_{k\alpha} w$$

and

$$0 = \partial_{k\alpha}\hat{w} \cdot \hat{w}^{-1} + \hat{w}E_\alpha z^k \hat{w}^{-1} - \overline{B}_{k\alpha}.$$

Taking the positive part, we obtain $\overline{B}_{k\alpha} = (\hat{w}E_\alpha z^k \hat{w}^{-1})_+ = (z^k R_\alpha)_+$. $\square$

Proposition 2. *Let $\hat{w}$ be a series $\hat{w} = \sum_{i_0}^{\infty} w_i z^{-i}$, $w_0 = I$ and $w = \hat{w} \exp \xi$. All the functions depend on variables $t_{k\alpha}$. Then if w satisfies the hierarchy equations (12) then the following bilinear identity*

$$\mathrm{res}_z z^i \partial_{k_1 \alpha_1} ... \partial_{k_s \alpha_s} w \cdot w^{-1} = 0 \tag{17}$$

holds for arbitrary sets of indices, $i \geq 0$.

Conversely, if there is another series $\hat{v} = \sum_{i_0}^{\infty} v_i z^{-i}$, $v_0 = I$, $v = \exp(-\xi)\hat{v}$ and

$$\mathrm{res}_z z^i \partial_{k_1 \alpha_1} ... \partial_{k_s \alpha_s} w \cdot v = 0$$

for all sets of indices then $v = w^{-1}$ and w is a Baker function of the hierarchy.

Proof. Let w be a Baker function of the hierarchy. Then, by virtue of the equation (12), the left-hand side of (12) is a residue of a polynomial which is zero.

Conversely, $\mathrm{res}_z z^i wv = 0$ for all i implies that $(wv)_- = 0$, $\hat{w}\hat{v} = I$, and $v = w^{-1}$. We have further

$$(\partial_{k\alpha} \hat{w} + (z^k R_\alpha)_- \hat{w})e^\xi = (\partial_{k\alpha} - (z^k R_\alpha)_+)w$$

where R_α is defined as in (11). Using the assumption, one gets

$$0 = \mathrm{res}_z z^i (\partial_{k\alpha} - (z^k R_\alpha)_+)w \cdot w^{-1} = \mathrm{res}_z z^i (\partial_{k\alpha} \hat{w} + (z^k R_\alpha)_- \hat{w})\hat{w}^{-1}.$$

This implies $\partial_{k\alpha} \hat{w} + (z^k R_\alpha)_- \hat{w} = 0$ which is the hierarchy equation (11). □

Proposition 3. *Baker functions of the mcKP are those of s-p ZS, more than that, the action of the operators $\partial_{k\alpha}$ on them is the same in both the hierarchies. In other words, the s-p ZS hierarchy is a restriction of the mcKP.*

Proof. The first statement follows from the fact that a Baker function of the s-p ZS hierarchy satisfies a bilinear identity (17) stronger than (1b). The converse part of the proposition of Sect.3 can be applied (letting $A^{-1}(w^a)^* = w^{-1}$). It also follows from the second statement. Let us prove the latter. Let A be an arbitrary constant diagonal matrix with distinct diagonal elements. Put $\partial = \sum_\alpha a_\alpha^{-1} \partial_{1\alpha}$. Let w be a Baker function of the single-pole ZS hierarchy. Then w satisfies Eq.(17) for every multi-index. It suffices to show that w satisfies the set of equations of the form $\partial_{k\alpha} w = \overline{B}_{k\alpha} w$ for all k and α where $\overline{B}_{k\alpha}$ are differential operators in ∂ (see Proposition, Sect.2). We have obvious relations:

$$\begin{aligned} \partial_{k\alpha} w &= (E_\alpha z^k + O(z^{k-1}))e^\xi, \\ A^q \partial^q w &= (z^q + O(z^{q-1}))e^\xi. \end{aligned}$$

whence $\partial_{k\alpha} w - E_\alpha A^k \partial^k w = O(z^{k-1}) \exp \xi = (V_{k-1} z^{k-1} + O(z^{k-2})) \exp \xi$. The process can be prolonged: $\partial_{k\alpha} w - E_\alpha A^k \partial^k w - V_{k-1} A^{k-1} \partial^{k-1} w = O(z^{k-2}) \exp \xi$ etc. In the end we have

$$\partial_{k\alpha} w - \overline{B}_{k\alpha} w \equiv \partial_{k\alpha} w - E_\alpha A^k \partial^k w - V_{k-1} A^{k-1} \partial^{k-1} w - ... - V_0 w = O(z^{-1})e^\xi$$

where $\overline{B}_{k\alpha}$ is a differential operator. Now, the bilinear identity

$$\mathrm{res}_z z^i (\partial_{k\alpha} - \overline{B}_{k\alpha})w \cdot w^{-1} = 0$$

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where $(\partial_{k\alpha} - \overline{B}_{k\alpha})w \cdot w^{-1} = O(z^{-1})$ implies that $(\partial_{k\alpha} - \overline{B}_{k\alpha})w \cdot w^{-1} = 0$, and $(\partial_{k\alpha} - \overline{B}_{k\alpha})w = 0$ as required. $\square$

Remark. It can seem strange that $\overline{B}_{k\alpha}w$ whose elements are differential polynomials with respect to ∂ coincides with $B_{k\alpha}$ where only ordinary polynomials are involved. The explanation is that one of equations of the s-p ZS hierarchy is $\partial w \equiv \sum a_{\alpha}^{-1}\partial_{1\alpha}w = B_{1\alpha}w$ or, in detail, $(\partial + [A^{-1}, w_1] - A^{-1}z)w = 0$. It enables us to eliminate all the derivatives. It is spectacular and instructive (though cumbersome) to verify the statement of the last proposition directly even in the simplest case of ∂_{2k}.

Corollary. *The τ-functions for the s-p ZS hierarchy exist and they are a special case of those for the mcKP.*

6. Not normalized s-p ZS hierarchy.

The hierarchy in the last section was normalized, in the sense that $w_0 = I$. Now w_0 also will depend on time variables, $w_0(t)$. The definition of the hierarchy (11) must be adjusted to this requirement since (11) implies that $w_0 =$ const.

Let $A_{(+)}$ symbolize the purely positive part of an expansion in powers of z, i.e., without the constant term, and $A_{(-)}$ negative part with the constant term, i.e. the constant term passes from the positive part to the negative one. Eq.(11) will be replaced by

$$\partial_{l\alpha}\hat{w} = -(z^l R_\alpha)_{(-)}\hat{w}, \quad R_\alpha = \hat{w}E_\alpha\hat{w}^{-1} \tag{18}$$

and Eq.(12) by

$$\partial_{l\alpha}w = B_{l\alpha} = (z^l R_a)_{(+)}.$$

It can be proven that the operators $\partial_{l\alpha}$ commute as well.

Proposition 1. *If $\hat{w}$ satisfies (18) then $\hat{v} = w_0^{-1}\hat{w}$ satisfies (11).*

Proof. Eq.(18) implies

$$\partial_{l\alpha}w_0 = -(z^l \hat{w}E_\alpha\hat{w}^{-1})_0 w_0 = -w_0(z^l \hat{v}E_\alpha\hat{v}^{-1})_0. \tag{19}$$

Then

$$\partial_{l\alpha}\hat{v} = -w_0^{-1}\partial_{l\alpha}w_0 \cdot w_0^{-1}\hat{w} + w_0^{-1}\partial_{l\alpha}\hat{w} =$$
$$= (z^l \hat{v}E_\alpha\hat{v}^{-1})_0\hat{v} - w_0^{-1}(z^l \hat{w}E_\alpha\hat{w}^{-1})_{(-)}\hat{w} = (z^l \hat{v}E_\alpha\hat{v}^{-1})_0\hat{v} - (z^l \hat{v}E_\alpha\hat{v}^{-1})_{(-)}\hat{v}$$
$$= -(z^l \hat{v}E_\alpha\hat{v}^{-1})_-\hat{v}.$$

This is exactly Eq.(11). $\square$

The proposition 1 allows to express $\hat{v}$ in terms of a τ function. However, this is not what we need, w_0 remains indefinite. In order to determine it one has to solve the linear equation (19). We show further that the whole function $\hat{w}$ has an expression in terms of τ-functions

$$\hat{w}_{\alpha\beta}(\zeta) = \frac{\tau_{\alpha\beta}(..., t_{k\gamma} - \delta_{\beta\gamma} \cdot 1/(k\zeta^k), ...)}{\tau(t)} \tag{20}$$

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for both $\alpha = \beta$ and $\alpha \neq \beta$.

First of all, one must write the bilinear identity. In this case, Eq.(17) holds in a stronger form: if $s > 0$ then (17) holds for $i \geq -1$, if $s = 0$ then it holds for $i \geq 0$ while $\mathrm{res}_z z^{-1} w \cdot w^{-1} = I$. The identity can be written in the dual form where all the derivatives act on w^{-1} rather than on w.

Proposition 2. *To every Baker function there exist functions τ and $\tau_{\alpha\beta}$ such that Eq.(20) holds up to an equivalence, i.e.,*

$$\hat{w}_{\alpha\beta}(\zeta) = c_\beta(\zeta) \frac{\tau_{\alpha\beta}(..., t_{k\gamma} - \delta_{\beta\gamma} \cdot 1/(k\zeta^k), ...)}{\tau(t)}$$

where $c_\beta(z)$ are constant series.

It follows from the bilinear identity (17) that

$$\mathrm{res}_z z^i w \cdot G_\beta(\zeta) w^{-1} = \begin{cases} 0, & \text{if } i \geq 0 \\ I, & \text{if } i = -1 \end{cases}$$

As in sect. 4 this transforms to

$$\mathrm{res}_z z^i \hat{w}(z)(I - E_\beta + (1 - \frac{z}{\zeta})^{-1} E_\beta) G_\beta(\zeta) \hat{w}^{-1} = \begin{cases} 0, & \text{if } i \geq 0 \\ I, & \text{if } i = -1 \end{cases}$$

We have

$$\mathrm{res}_z z^i \hat{w}(z)(I - E_\beta) G_\beta(\zeta) \hat{w}^{-1}(z)$$

$$+\zeta(z^i \hat{w}(z) G_\beta(\zeta) E_\beta \hat{w}^{-1}(z))_- |_{z=\zeta} = \begin{cases} 0, & \text{if } i \geq 0 \\ I, & \text{if } i = -1 \end{cases}$$

Let $i = -1$. Then this equality becomes

$$w_0(I - E_\beta) G_\beta(\zeta) w_0^{-1} + \hat{w}(\zeta) G_\beta(\zeta) E_\beta \hat{w}^{-1}(\zeta) = I$$

or, multiplying by $G_\beta w_0$,

$$w_0(I - E_\beta) + \hat{w}(\zeta) G_\beta(\zeta) E_\beta \hat{w}^{-1}(\zeta) w_0 = G_\beta(\zeta) w_0.$$

For the (α, β)th element this is

$$\hat{w}_{\alpha\beta} G_\beta(\hat{v}^{-1})_{\beta\beta} = G_\beta(w_0)_{\alpha\beta}. \tag{21}$$

It is easy to see that the case $i = 0$ gives the same for $\hat{v}$ as it was in sect. 4 for $\hat{w}$; in particular, the analogues of (2) and (3), and the possibility to express $\hat{v}$ in terms of a τ-function. Eq.(2) becomes

$$G_\beta(\hat{v}^{-1})_{\beta\beta} = (\hat{v}_{\beta\beta})^{-1}. \tag{22}$$

We even do not need to prove this since we knew this in advance. We know also that there is a function $\tau(t)$ and constant series c_β such that $\hat{v}_{\beta\beta} = c_\beta G_\beta \tau \cdot \tau^{-1}$. With the help of Eq.(22), Eq.(21) transforms to

$$\hat{w}_{\alpha\beta}(\hat{v}_{\beta\beta})^{-1} = G_\beta(w_0)_{\alpha\beta}. \tag{23}$$

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Now, let

$$\tau_{\alpha\beta} = \tau(w_0)_{\alpha\beta}. \tag{24}$$

Using (23) and (22), we have

$$\frac{G_\beta \tau_{\alpha\beta}}{\tau} = \frac{G_\beta \tau}{\tau} \cdot G_\beta(w_0)_{\alpha\beta} = c_\beta^{-1} \hat{v}_{\beta\beta} \hat{w}_{\alpha\beta} (\hat{v}_{\beta\beta})^{-1} = c_\beta^{-1} \hat{w}_{\alpha\beta}$$

as required. $\square$

Notice, that Eqs.(22) and (23) look very nice being put together in the form

$$\begin{aligned}
G_\beta(\hat{v}^{-1})_{\beta\beta} &= (\hat{v}_{\beta\beta})^{-1}, \\
G_\beta(\hat{w}\hat{v}^{-1})_{\alpha\beta} &= \hat{w}_{\alpha\beta}(\hat{v}_{\beta\beta})^{-1}.
\end{aligned}$$

7. The general ZS hierarchy.

This hierarchy was introduced in [7]. Let a_k, $k = 1, ..., m$ be a given set of complex numbers. Let, for every k,

$$\hat{w}_k = \sum_0^\infty w_{ki}(z - a_k)^i,$$

be a formal series. The entries of $n \times n$ matrices w_{ki}, $w_{ki,\alpha\beta}$ are just letters. We consider the algebra $\mathcal{A}_w$ of polynomials of all this entries and $(\det w_{k0})^{-1}$. The formal series $\hat{w}_k$ can be inverted within this algebra. Let

$$R_{k\alpha} = \hat{w}_k E_\alpha \hat{w}_k^{-1}; \quad R_{k\alpha l} = R_{k\alpha}(z - a_k)^{-l}$$

where E_α is, as before, a matrix with only one non-vanishing element, equal 1, on the (α, α) place.

We have the following objects. Such quantities as $\hat{w}_k$ and $R_{k\alpha l}$ are formal series, or jets, at the points a_k. The algebra of all such jets will be called J_k and $J = \oplus J_k$. If $j_k \in J_k$ is a jet then j_k^- symbolizes its principal part, i.e., a sum of negative powers of $z - a_k$, and j_k^+ the rest of the series. Correspondingly, the jet algebras split into parts, $J_k = J_k^+ \oplus J_k^-$. If the principal part contains finite number of terms (and we tacitly assume this unless the opposite is said or is evident from a context) it can be considered as a global meromorphic function; the algebra of global meromorphic functions is G. A global function gives rise to a jet at every a_k. In particular, j_k^- can be considered as a jet at a point a_{k_1}, different from a_k, more precisely, as an element of $J_{k_1}^+$. And finally, there will be formal products of jets or of global functions by expressions of the form $\exp \xi_k$ where

$$\xi_k = \sum_{\alpha=1}^n \sum_{l=0}^\infty t_{k\alpha l} E_\alpha (z - a_k)^{-l}.$$

Definitions. (i) *A hierarchy corresponding to a fixed set $\{a_k\}$ is the totality of equations*

$$\partial_{k\alpha l} \hat{w}_{k_1} = \begin{cases} -R_{k\alpha l}^+ \hat{w}_{k_1}, & k = k_1 \\ R_{k\alpha l}^- \hat{w}_{k_1}, & \text{otherwise} \end{cases}, \quad \partial_{k\alpha l} = \partial/\partial t_{k\alpha l}. \tag{25}$$

In the second case $R_{k\alpha l}^-$ is considered as an element of $J_{k_1}^+$, see above; $t_{k\alpha l}$ are some variables.

(ii) *A ZS hierarchy is an inductive limit of hierarchies with fixed sets $\{a_k\}$, with respect to a natural embedding of a hierarchy corresponding to a subset into a hierarchy corresponding to a larger set, as a subhierarchy.*

In this article we deal with the hierarchy corresponding to a fixed set $\{a_k\}$. There was proven in [7] that all the equations of the hierarchy commute. The following proposition readily can be checked by a simple straightforward computation:

Proposition 1. *A dressing formula*

$$\hat{w}_{k_1}(\partial_{k\alpha l} - E_\alpha(z - a_k)^{-l}\delta_{kk_1})\hat{w}_{k_1}^{-1} = \partial_{k\alpha l} - B_{k\alpha l}, \quad B_{k\alpha l} = R_{k\alpha l}^{-} \tag{26}$$

is equivalent to Eq.(25).

The operator $\partial_{k\alpha l} - B_{k\alpha l}$ is assumed to act in J_{k_1}. However, it does not depend on k_1 at all and can be considered as a global function of z with the only pole of the lth order at a_k.
    Let

$$w_k = \hat{w}_k \exp \xi_k.$$

Definition. *The collection $w = \{w_k\}$ is the formal Baker function of the hierarchy.*

Eq.(25) can be written in terms of the Baker function as

$$\partial_{k\alpha l}w_{k_1} = B_{k\alpha l}w_{k_1} \tag{27}$$

and Eq.(26) as

$$w_{k_1}\partial_{k\alpha l}w_{k_1}^{-1} = \partial_{k\alpha l} - B_{k\alpha l}. \tag{28}$$

Proposition 2. *All the operators $\partial_{k\alpha l} - B_{k\alpha l}$ commute.*

Proof. This is a corollary of the fact that $\partial_{k\alpha l}$ commute and Eq. (28). $\square$

One can consider arbitrary linear combinations of the above constructed operators,

$$L = \sum_{k,\alpha,l} \lambda_{k\alpha l}(\partial_{k\alpha l} - B_{k\alpha l}) = \partial + U$$

where $\partial = \sum_{k,\alpha,l} \lambda_{k\alpha l}\partial_{k\alpha l}$ and $U = -\sum_{k,\alpha,l} \lambda_{k\alpha l}B_{k\alpha l}$. Two such operators commute which yields equations of the Zakharov-Shabat type

$$\partial U_1 - \partial_1 U = [U_1, U].$$

Functions U and U_1 are rational functions of the parameter z.

Remark 1. A Baker function is determined up to an equivalence. Two Baker functions $w^{(1)}$ and $w^{(2)}$ are equivalent if there are constant diagonal matrices $c_k(z) = \sum_0^\infty c_{ki}z^{-i} = \text{diag}$ $(c_{k\beta}(z))$ such that $w_k^{(1)} = w_k^{(2)}c_k$, i.e., $w_{k,\alpha\beta}^{(1)} = c_{k,\beta}w_{k,\alpha\beta}^{(2)}$. Equivalent Baker functions generate

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the same Lax operator L.

Remark 2. Here we have a special case of ZS equation: the functions U and U_1 vanishing at infinity. If we make a gauge transformation $w_k \mapsto g(t)w_k$ then $\partial + U \mapsto g(\partial + U)g^{-1} = \partial + gUg^{-1} - (\partial g)g^{-1}$, the last term does not vanish at infinity. This yields the general case.

If we deal with only one component w_k of the Baker function, and consider its dependence solely on the variables $t_{k\alpha l}$ with the same k (local variables) ignoring all the others (alien variables), e.g., fixing their values as parameters then we shall have a single-pole non-normalized hierarchy in the sense of the previous section. (One has to perform a transformation $(z - a_k)^{-1} = \zeta$). This fact allows to apply all the formulas obtained in that section to the present case. In particular, there are functions $\tau_k(t)$ and $\tau_{k\alpha\beta}(t)$ depending on local as well as on alien variables such that

$$w_{k,\alpha\beta}(t, z) = c_{k\beta}(z)\frac{G_{k\beta}(z)\tau_{k,\alpha\beta}(t)}{\tau_k(t)}e^{\xi_k} \tag{29}$$

where operators of translation $G_{k\beta}(z)$ are defined by

$$G_{k\beta}f(t) = f(..., t_{k_1,\gamma,l} - \delta_{kk_1}\delta_{\beta\gamma}\frac{1}{l}(z - a_k)^l, ...)$$

and $c_{k\beta}(z)$ are constant series in $z - a_k$.

We have not used yet the equations of the hierarchy with respect to the alien variables. The rest of the section will be devoted to the proof that if those equations are taken into account, then, roughly speaking, all denominators τ_k in the previous formula are equal. More precisely, the following theorem holds:

Theorem. *If $w = \{w_k\}$ is an arbitrary Baker function then there are functions $\tau(t)$ and $\tau_{k,\alpha\beta}(t)$ and constant series $c_{k\beta}(z)$ such that*

$$w_{k,\alpha\beta}(t, z) = c_{k\beta}(z)\frac{G_{k\beta}(z)\tau_{k,\alpha\beta}(t)}{\tau(t)}e^{\sum_l \xi_l}.$$

Notice that the last factor is $\exp\sum_l \xi_l$ and not just $\exp\xi_k$, therefore the expression in front of it is not $\hat{w}_k$. We call it $\hat{\hat{w}}_k$. Thus,

$$w_k = \hat{\hat{w}}_k \exp\sum_l \xi_l, \quad \hat{w}_k = \hat{\hat{w}}_k \exp\sum_{l \neq k} \xi_l, \quad \hat{w}_{k0} = \hat{\hat{w}}_{k0} \exp\sum_{l \neq k} \xi_l(a_k).$$

Proof. We already have Baker functions w_k, $\hat{w}_k = w_k \exp(-\xi_k)$ and $\hat{\hat{w}}_k = \hat{w}_k \exp(-\sum_{l \neq k}\xi_l)$. Let us also introduce, as we did in Sect.6,

$$v_k(z) = w_{k0}^{-1}w_k(z), \quad \hat{v}_k(z) = v_k(z)\exp(-\xi_k) = w_{k0}^{-1}\hat{w}_k(z)$$

and

$$\hat{\hat{v}}_k(z) = \hat{\hat{w}}_{k0}^{-1}\hat{\hat{w}}_k(z) = \exp\sum_{l \neq k}\xi_l(a_k)\hat{v}_k(z)\exp(-\sum_{l \neq k}\xi_l).$$

What is important, $\hat{v}_k$ and $\hat{\hat{v}}_k$ differ by two diagonal factors, on the left and on the right which do not depend on the variables with the same index k, the local variables. The series $\hat{w}_k$ and $\hat{\hat{w}}_k$ differ by a right factor of the same kind.

Considering w_k as a function of local variables, we have noticed that this is a Baker function of a single-pole not normalized hierarchy, and $v_k(z)$ that of the corresponding normalized hierarchy. Therefore, one can write for them Eq.(22), or in present notations,

$$G_{k\beta}(\zeta)(\hat{v}_k^{-1}(\zeta))_{\beta\beta} = (\hat{v}_{k,\beta\beta}(\zeta))^{-1} \tag{30}$$

and Eq.(23), or

$$\hat{w}_{k,\alpha\beta}(\zeta)(\hat{v}_{k,\beta\beta})^{-1} = G_{k\beta}(\zeta)w_{k0,\alpha\beta}. \tag{31}$$

The same equations can be written for $\hat{\hat{v}}_k$ and $\hat{\hat{w}}_{k0}$ since the diagonal factors we discussed above will cancel. They do not depend on local variables and the operators $G_{k\beta}$ do not act on them. Thus,

$$G_{k\beta}(\zeta)(\hat{\hat{v}}_k^{-1}(\zeta))_{\beta\beta} = (\hat{\hat{v}}_{k,\beta\beta}(\zeta))^{-1} \tag{30$'$}$$

and

$$\hat{\hat{w}}_{k,\alpha\beta}(\zeta)(\hat{\hat{v}}_{k,\beta\beta})^{-1} = G_{k\beta}\hat{\hat{w}}_{k0,\alpha\beta}. \tag{31$'$}$$

By the same reason we have the formula

$$\frac{G_{k\beta_2}(\zeta_2)\hat{\hat{v}}_{k,\beta_1\beta_1}(\zeta_1)}{\hat{\hat{v}}_{k,\beta_1\beta_1}(\zeta_1)} = \frac{G_{k\beta_1}(\zeta_1)\hat{\hat{v}}_{k,\beta_2\beta_2}(\zeta_2)}{\hat{\hat{v}}_{k,\beta_2\beta_2}(\zeta_2)}. \tag{32}$$

It is correct for $\hat{v}_k$ since this is, virtually, Eq.(5). The additional diagonal factors cancel, so this is also correct for $\hat{\hat{v}}_k$.

Lemma 1. *The equality*

$$\frac{G_{k_2\beta}(\zeta_2)\hat{\hat{v}}_{k_1,\beta\beta}(\zeta_1)}{\hat{\hat{v}}_{k_1,\beta\beta}(\zeta_1)} = \frac{G_{k_1\beta}(\zeta_1)\hat{\hat{v}}_{k_2,\beta\beta}(\zeta_2)}{\hat{\hat{v}}_{k_2,\beta\beta}(\zeta_2)} \tag{33}$$

holds.

Proof of the lemma 1. Eq.(27) implies that $\partial_{k_1\alpha l}w_k \cdot w_k^{-1} = B_{k_1\alpha l} = R_{k_1\alpha l}^-$ is a meromorphic function with a single pole at a_{k_1}, vanishing at infinity and *not depending on k*. Actually, it is easy to see that this is a characteristic property of the hierarchy which expresses its universality, but we do not use this fact below. More generally, $\partial_{k_1\alpha_1 l}...\partial_{k_s\alpha_s l}w_k \cdot w_k^{-1}$ does not depend on k and is a meromorphic function with the poles at $a_{k_1},...,a_{k_s}$ vanishing at infinity when $s > 0$. The same is also true for $w_k \cdot \partial_{k_1\alpha_1 l}...\partial_{k_s\alpha_s l}w_k^{-1}$. This implies that the expression $(z - a_{k_1})^{-1}(z - a_{k_2})^{-1}w_k G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)w_k^{-1}$ is a meromorphic function having the only poles on the Riemann sphere at a_{k_1} and a_{k_2} and not depending on k. The sum of residues must vanish. Computing the residue at a_{k_1} we replace k by k_1 and doing this at a_{k_2} we replace k by k_2. For simplicity of writing, let res_{k_i} symbolize $\mathrm{res}_{a_{k_i}}$. We have

$$\mathrm{res}_{k_1}(z - a_{k_1})^{-1}(z - a_{k_2})^{-1}w_{k_1}(z)G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)w_{k_1}^{-1}(z)$$

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$$+\mathrm{res}_{k_2}(z-a_{k_1})^{-1}(z-a_{k_2})^{-1}w_{k_2}(z)G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)w_{k_2}^{-1}(z)=0.$$

In terms of $\hat{w}_k$ this identity can be written as

$$\mathrm{res}_{k_1}(z-a_{k_1})^{-1}(z-a_{k_2})^{-1}\hat{w}_{k_1}(z)(I-E_\beta+E_\beta(1-\frac{\zeta_1-a_{k_1}}{z-a_{k_1}})^{-1})$$

$$\cdot(I-E_\beta+E_\beta(1-\frac{\zeta_2-a_{k_2}}{z-a_{k_2}})^{-1})G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\hat{w}_{k_1}^{-1}(z)+(k_1,\zeta_1\Leftrightarrow k_2,\zeta_2)=0,$$

i.e.,

$$\mathrm{res}_{k_1}(z-a_{k_1})^{-1}(z-a_{k_2})^{-1}\hat{w}_{k_1}(z)(I-E_\beta+E_\beta(1-\frac{\zeta_1-a_{k_1}}{z-a_{k_1}})^{-1}$$

$$\cdot(1-\frac{\zeta_2-a_{k_2}}{z-a_{k_2}})^{-1})G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\hat{w}_{k_1}^{-1}(z)+(k_1,\zeta_1\Leftrightarrow k_2,\zeta_2)=0,\qquad(34)$$

Here $(k_1,\zeta_1\Leftrightarrow k_2,\zeta_2)$ denotes a term obtained by switching k_1 and k_2, ζ_1 and ζ_2. In the previous sections we computed similar residues several times, so it does not need much explanation. The term with $I-E_\beta$ gives

$$(a_{k_1}-a_{k_2})\hat{w}_{k_10}(I-E_\beta)G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\hat{w}_{k_10}^{-1}+(k_1,\zeta_1\Leftrightarrow k_2,\zeta_2).$$

Two others are

$$(\zeta_1-a_{k_2})^{-1}\hat{w}_{k_1}(\zeta_1)E_\beta(1-\frac{\zeta_2-a_{k_2}}{\zeta_1-a_{k_2}})^{-1}G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\hat{w}_{k_1}^{-1}(\zeta_1)+(k_1,\zeta_1\Leftrightarrow k_2,\zeta_2)$$

$$=(\zeta_1-\zeta_2)^{-1}[\hat{w}_{k_1}(\zeta_1)E_\beta G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\hat{w}_{k_1}^{-1}(\zeta_1)-(k_1,\zeta_1\Leftrightarrow k_2,\zeta_2)].$$

Multiplying thus transformed Eq.(34) by $\hat{w}_{k_10}^{-1}$ on the left and by $G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\hat{w}_{k_20}$ on the right we obtain

$$(a_{k_1}-a_{k_2})[(I-E_\beta)G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\hat{w}_{k_10}^{-1}\hat{w}_{k_20}-\hat{w}_{k_10}^{-1}\hat{w}_{k_20}(I-E_\beta)]$$

$$+(\zeta_1-\zeta_2)^{-1}[\hat{v}_{k_1}(\zeta_1)E_\beta G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\hat{v}_{k_1}^{-1}(\zeta_1)\hat{w}_{k_10}^{-1}\hat{w}_{k_20}$$

$$-\hat{w}_{k_10}^{-1}\hat{w}_{k_20}\hat{v}_{k_2}(\zeta_2)E_\beta G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\hat{v}_{k_2}^{-1}(\zeta_2)]=0$$

where $\hat{v}_k(\zeta)=\hat{w}_{k0}^{-1}\hat{w}_k(\zeta)$. Now let us take the (β,β)th element of this identity. The first two terms are not involved in it, by virtue of the factors $I-E_\beta$. Two others yield

$$\hat{v}_{k_1,\beta\beta}G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)(\hat{v}_{k_1}^{-1}(\zeta)T_{k_1k_2})_{\beta\beta}$$

$$=(T_{k_1k_2}\hat{v}_{k_2}(\zeta_2))_{\beta\beta}G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)(\hat{v}_{k_2}^{-1}(\zeta_2))_{\beta\beta}\qquad(35)$$

where

$$T_{k_1k_2}=\hat{w}_{k_10}^{-1}\hat{w}_{k_20},$$

the transition function. Using Eq.(30'), we can replace $G_{k_2\beta}(\zeta_2)(\hat{\hat{v}}_{k_2}^{-1}(\zeta_2))_{\beta\beta}$ by $(\hat{\hat{v}}_{k_2,\beta\beta}(\zeta_2))^{-1}$.
Eq.(35) becomes

$$\hat{\hat{v}}_{k,\beta\beta}G_{k_1\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)(\hat{\hat{v}}_{k_1}^{-1}(\zeta)T_{k_1k_2})_{\beta\beta}$$

$$= (T_{k_1k_2}\hat{\hat{v}}_{k_2}(\zeta_2))_{\beta\beta}G_{k_1\beta}(\zeta_1)(\hat{\hat{v}}_{k_2,\beta\beta}(\zeta_2))^{-1}. \tag{36}$$

Let $\zeta_1 = a_{k_1}$. Then Eq.(36) becomes

$$G_{k_2\beta}(\zeta_2)T_{k_1k_2,\beta\beta} = (T_{k_1k_2}\hat{\hat{v}}_{k_2}(\zeta_2))_{\beta\beta}(\hat{\hat{v}}_{k_2,\beta\beta}(\zeta_2))^{-1}.$$

This enables us to rewrite the right-hand side of (36) as

$$\hat{\hat{v}}_{k_2,\beta\beta}(\zeta_2)\frac{G_{k_2\beta}(\zeta_2)T_{k_1k_2,\beta\beta}}{G_{k_1\beta}(\zeta_1)\hat{\hat{v}}_{k_2,\beta\beta}(\zeta_2)}.$$

Now, let $\zeta_2 = a_{k_2}$. Eq.(36) transforms to

$$\hat{\hat{v}}_{k_1,\beta\beta}(\zeta_1)G_{k_1\beta}(\zeta_1)(\hat{\hat{v}}_{k_1}^{-1}(\zeta)T_{k_1k_2})_{\beta\beta} = T_{k_1k_2,\beta\beta}.$$

The left-hand side of (36) can be written as

$$\hat{\hat{v}}_{k_1,\beta\beta}(\zeta_1)G_{k_2\beta}(\zeta_2)\frac{T_{k_1k_2,\beta\beta}}{\hat{\hat{v}}_{k_1,\beta\beta}(\zeta_1)}.$$

Equating the left- and the right-hand sides and cancelling the common factor $G_{k_2\beta}(\zeta_2)T_{k_1k_2,\beta\beta}$, we obtain the required identity (33). $\square$

Lemma 2. *The equation*

$$\frac{G_{k_2\beta_2}(\zeta_2)\hat{\hat{v}}_{k_1,\beta_1\beta_1}(\zeta_1)}{\hat{\hat{v}}_{k_1,\beta_1\beta_1}(\zeta_1)} = \frac{G_{k_1\beta_1}(\zeta_1)\hat{\hat{v}}_{k_2,\beta_2\beta_2}(\zeta_2)}{\hat{\hat{v}}_{k_2,\beta_2\beta_2}(\zeta_2)} \tag{37}$$

holds for any k_1, k_2, β_1 and β_2.

Proof of the lemma 2. We already have two special cases of this lemma: Eq.(32) for $k_1 = k_2$ and lemma 1 for $\beta_1 = \beta_2$. Now, suppose neither of these conditions holds. Similarly to what we did proving the lemma 1, we write a bilinear identity

$$\mathrm{res}_{k_1}(z - a_{k_1})^{-1}(z - a_{k_2})^{-1}w_{k_1}(z)G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)w_{k_1}^{-1}(z) + (k_1, \zeta_1, \beta_1 \Leftrightarrow k_2, \zeta_2, \beta_2) = 0.$$

In terms of $\hat{w}_k$ this identity can be written as

$$\mathrm{res}_{k_1}(z - a_{k_1})^{-1}(z - a_{k_2})^{-1}\hat{w}_{k_1}(z)(I - E_{\beta_1} + E_{\beta_1}(1 - \frac{\zeta_1 - a_{k_1}}{z - a_{k_1}})^{-1})$$

$$\cdot(I - E_{\beta_2} + E_{\beta_2}(1 - \frac{\zeta_2 - a_{k_2}}{z - a_{k_2}})^{-1})G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)\hat{w}_{k_1}^{-1}(z) + (k_1, \zeta_1, \beta_1 \Leftrightarrow k_2, \zeta_2, \beta_2) = 0,$$

i.e.,

$$\text{res}_{k_1}(z - a_{k_1})^{-1}(z - a_{k_2})^{-1}\hat{\hat{w}}_{k_1}(z)(I - E_{\beta_1} - E_{\beta_2} + E_{\beta_1}(1 - \frac{\zeta_1 - a_{k_1}}{z - a_{k_1}})^{-1}$$

$$+E_{\beta_2}(1 - \frac{\zeta_2 - a_{k_2}}{z - a_{k_2}})^{-1})G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)\hat{\hat{w}}_{k_1}^{-1}(z) + (k_1, \zeta_1, \beta_1 \Leftrightarrow k_2, \zeta_2, \beta_2) = 0.$$

Computing the residues, we have

$$(a_{k_1} - a_{k_2})^{-1}\hat{\hat{w}}_{k_10}(I - E_{\beta_1} - E_{\beta_2})G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)\hat{\hat{w}}_{k_10}^{-1}$$

$$(\zeta_1 - a_{k_2})^{-1}\hat{\hat{w}}_{k_1}(\zeta_1)E_{\beta_1}G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)\hat{\hat{w}}_{k_1}^{-1}(\zeta_1)$$

$$+(a_{k_1} - a_{k_2})^{-1}((1 - \frac{\zeta_2 - a_{k_2}}{a_{k_1} - a_{k_2}})^{-1}\hat{\hat{w}}_{k_10}^{-1} + (k_1, \zeta_1, \beta_1 \Leftrightarrow k_2, \zeta_2, \beta_2) = 0.$$

Dividing by $\hat{\hat{w}}_{k_10}$ on the left, by $G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)\hat{\hat{w}}_{k_20}^{-1}$ on the right, we have

$$*(I - E_{\beta_1} - E_{\beta_2}) + (I - E_{\beta_1} - E_{\beta_2}) * +(\zeta_1 - a_{k_2})^{-1}\hat{\hat{v}}_{k_1}(\zeta_1)E_{\beta_1}G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)\hat{\hat{v}}_{k_1}^{-1}(\zeta_1)T_{k_1k_2}$$

$$+(\zeta_2 - a_{k_1}^{-1}T_{k_1k_2}\hat{\hat{v}}_{k_2}(\zeta_2)E_{\beta_2}G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)\hat{v}_{k_2}^{-1} - E_{\beta_2} * + * E_{\beta_1} = 0$$

where asterisks symbolize various factors which are not written in detail since they are not important below.

Take the (β_1, β_2)th element of this equality. The terms with asterisks vanish. The following terms remain:

$$(\zeta_1 - a_{k_2})^{-1}\hat{\hat{v}}_{k_1\beta_1\beta_1}(\zeta_1)G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)(\hat{\hat{v}}_{k_1}^{-1}(\zeta_1)T_{k_1k_2})_{\beta_1\beta_1}$$

$$+(\zeta_2 - a_{k_1})^{-1}(T_{k_1k_2}\hat{\hat{v}}_{k_2}(\zeta_2))_{\beta_1\beta_2}G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)(\hat{v}_{k_2}^{-1}(\zeta_2))_{\beta_2\beta_2} = 0. \tag{38}$$

Now, let $\zeta_1 = a_{k_1}$:

$$(a_{k_1} - a_{k_2})^{-1}G_{k_2\beta_2}(\zeta_2)(T_{k_1k_2})_{\beta_1\beta_2}$$

$$+(\zeta_2 - a_{k_1})^{-1}(T_{k_1k_2}\hat{\hat{v}}_{k_2}(\zeta_2))_{\beta_1\beta_2}G_{k_2\beta_2}(\zeta_2)(\hat{v}_{k_2}^{-1})_{\beta_2\beta_2} = 0. \tag{39}$$

Using this equality, transform the second term of (38):

$$-\frac{(a_{k_1} - a_{k_2})^{-1}G_{k_2\beta_2}(\zeta_2)(T_{k_1k_2})_{\beta_1\beta_2}G_{k_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2)(\hat{v}_{k_2}^{-1}(\zeta_2))_{\beta_2\beta_2}}{G_{k_2\beta_2}(\zeta_2)(\hat{v}_{k_2}^{-1})_{\beta_2\beta_2}}$$

$$= -\frac{(a_{k_1} - a_{k_2})^{-1}G_{k_2\beta_2}(\zeta_2)(T_{k_1k_2})_{\beta_1\beta_2}\hat{\hat{v}}_{k_2,\beta_2\beta_2}(\zeta_2)}{G_{k_1\beta_1}(\zeta_1)\hat{\hat{v}}_{k_2,\beta_2\beta_2}(\zeta_2)}.$$

(We have used Eq.(30') doing the last transformation).

Let $\zeta_2 = a_{k_2}$:

$$(\zeta_1 - a_{k_2})^{-1}\hat{\hat{v}}_{k_1\beta_1\beta_1}(\zeta_1)G_{k_1\beta_1}(\zeta_1)(\hat{\hat{v}}_{k_1}^{-1}(\zeta_1)T_{k_1k_2})_{\beta_1\beta_1} + (a_{k_1} - a_{k_2})^{-1}(T_{k_1k_2})_{\beta_1\beta_2} = 0 \tag{40}$$

whence the first term can be written as

$$-(a_{k_1} - a_{k_2})^{-1}\hat{\hat{v}}_{k_1\beta_1\beta_1}(\zeta_1)G_{k_2\beta_2}(\zeta_2) \cdot \frac{(T_{k_1k_2})_{\beta_1\beta_2}}{\hat{\hat{v}}_{k_1\beta_1\beta_1}(\zeta_1)}.$$

The identity (38) becomes, after a cancelation of the common factor,

$$\frac{\hat{\hat{v}}_{k_2,\beta_2\beta_2}(\zeta_2)}{G_{k_1\beta_1}(\zeta_1)\hat{\hat{v}}_{k_2,\beta_2\beta_2}(\zeta_2)} = \frac{\hat{\hat{v}}_{k_1\beta_1\beta_1}(\zeta_1)}{G_{k_2\beta_2}(\zeta_2)\hat{\hat{v}}_{k_1\beta_1\beta_1}(\zeta_1)}$$

which is the statement of the lemma. □

Lemma 3. *The equation (37) implies that there is a function $\tau(t)$ and constant series $c_{k\beta}(\zeta)$ in powers of $\zeta - a_k$ such that $\hat{\hat{v}}_{k,\beta\beta}(\zeta) = c_{k\beta}G_{k\beta}\tau \cdot \tau^{-1}$.*

Proof of the lemma 3. Taking the logarithm of (37) and denoting $\ln\hat{\hat{v}}_{k,\beta\beta} = f_{k,\beta\beta}$ we have

$$(G_{k_1\beta_1}(\zeta_1) - 1)f_{k_2\beta_2\beta_2}(\zeta_2) = (G_{k_2\beta_2}(\zeta_2) - 1)f_{k_1\beta_1\beta_1}(\zeta_1),$$

and we just have to repeat the derivation of the Eq.(8) from (5) in Sect.4. □

The end of the proof of the theorem. Put $\tau_{k,\alpha\beta} = \tau \cdot \hat{w}_{k0\alpha\beta}$. Taking into account (31'), we have

$$\frac{G_{k\beta}(\zeta)\tau_{k,\alpha\beta}}{\tau} = \frac{G_{k\beta}(\zeta)\tau}{\tau} \cdot G_{k\beta}(\zeta)\hat{w}_{k0,\alpha\beta} = c_{k\beta}^{-1}\hat{\hat{v}}_{k,\beta\beta} \cdot \hat{w}_{k,\alpha\beta}\hat{\hat{v}}_{k,\beta\beta}^{-1} = c_{k\beta}^{-1}\hat{w}_{k,\alpha\beta}$$

as required. □

References.

1. Sato, M.: Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, RIMS Kokyuroku, 439, 30-46, 1981.

2. Date, E., Jimbo, M., Kashiwara, M., and Miwa, T.: Transformation groups for soliton equations, in: Jimbo and Miwa (ed.) Non-linear integrable systems – classical theory and quantum theory, Proc. RIMS symposium, Singapore, 1983.

3. Date, E., Jimbo, M., Kashiwara, M., and Miwa, T.: Operator approach to the Kadomtsev-Petviashvili equation - transformation groups for soliton equations III, Journ. Phys. Soc. Japan, 50, 3806-3812, 1981.

4. Ueno, K., and Takasaki, K.: Toda lattice hierarchy, in: Advanced Studies in Pure Mathematics, 4, World Scientific, 1-95, 1984.

5. Dickey, L. A.: On Segal-Wilson's definition of the τ-function and hierarchies AKNS-D and mcKP, in: Integrable systems, The Verdier Memorial Conference, Birkhäuser, 147-162, 1993.

6. Dickey, L. A.: On the τ-function of matrix hierarchies of integrable equations, Journal Math. Physics, 32, 2996-3002, 1991.

7. Dickey, L. A.: Why the general Zakharov-Shabat equations form a hierarchy, Com. Math. Phys., 163, 509-521, 1994.

8. Vasilev, S.: Tau functions of algebraic geometrical solutions to the general Zakharov-Shabat hierarchy, Preprint of the University of Oklahoma, 1994.

9. Dickey, L. A.: Soliton Equations and Hamiltonian Systems, Advanced Series in Mathematical Physics, 12, World Scientific, 1991.